New Form of Strip Approximation*

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A detailed set of "bootstrap" equations is formulated for zero-spin "external" particles based on a combination of the N/D method with the superposition of top-ranking Regge poles in all three reactions of a four-line connected part. The contribution from each pole arises from a distinct strip in the Mandelstam representation so that double counting is avoided. Only real values of l with $l\leqslant 1$ need be considered in the bootstrap calculation. The amplitude emerging from our N/D equations is meromorphic in the right-half l plane, and the Regge poles approach high-energy limits that are dynamically determined and which in some cases may lie to the right of $l=0$. The reduced residues vanish in the high-energy limit.

I. INTRODUCTION

It has been proposed that an approximation procedure for strong-interaction "bootstrap" calculations might be based on a combination of the N/D method with the superposition of a finite number of top-ranking Regge poles for *all* the different channels connected by analytic continuation.[1] By "top-ranking" is meant poles whose trajectories reach or closely approach the right-half l plane for real values of the energy. Since it is expected that these leading poles make large contributions over only a finite energy interval (at most a few GeV in width), the approximation is designed to be accurate in "strips" covering the low-energy resonance region and high energies at low momentum transfer. The spirit of our scheme is similar in this sense to that of the strip approximation proposed earlier by Chew and Frautschi[2] but differs through its dependence on continuation in angular momentum with the consequent absence of arbitrary coupling constants. The first paper dealing with the Regge-strip approximation[1] contains at least one mathematical error and certain of the assumptions need re-examination. In this paper we present a revised set of strip equations and analyze certain general features of their solutions.

Physically the most significant features relate to the asymptotic behavior of pole positions and residues. The poles generated by our N/D equations do not necessarily all retreat to the left-half J plane but their reduced residues decrease with a negative power of energy outside the strip. It is this behavior of the residues that is primarily responsible for the dominance of the strip regions.

II. THE SUPERPOSITION OF POLE CONTRIBUTIONS

The Mandelstam representation breaks the two-body scattering amplitude into three portions corresponding to the three possible pairings of the channel variables s, t, u. For example, the (s,t) portion is[3]

$$A^{st}(s,t)=\frac{1}{\pi^2}\int\int ds'dt'\frac{\rho(s',t')}{(s'-s)(t'-t)}, \qquad \text{(II.1)}$$

where subtractions if necessary are to be determined by analytic continuation from large l in the s and t channels. Explicitly, if one assumes an analytic interpolation between all physical l values as well as meromorphy in the right-half angular-momentum plane, $A^{st}(s,t)$ may be decomposed into three parts[4]:

$$A^{st}(s,t)=\frac{1}{\pi^2}\int_{s_0}^{\infty}\frac{ds'}{s'-s}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\rho_{st}'(s',t')+\sum_i R_i^{t_1}(s,t)+\sum_j R_j^{s_1}(t,s), \qquad \text{(II.2)}$$

where the first term needs no subtractions and the second and third arise from Regge poles in the s and t channels, respectively. Mathematically speaking, only poles that reach the right-half l plane for some real interval of energy need be recognized; the remaining poles may remain buried in the first term of (II.2). It is proposed here, however, also to separate out any poles that closely approach the right-half l plane in order to make the remainder as small as possible.

Assuming all particles of the same mass, we shall take the following formula for the contribution from the ith pole in the s channel:

$$R_i^{t_1}(s,t)=\frac{1}{\pi}\int_{t_1}^{\infty}\frac{R_i(t',s)}{t'-t}dt', \qquad \text{(II.3)}$$

with

$$R_i(t,s)=\frac{\pi}{2}[2\alpha_i(s)+1]\gamma_i(s)(-q_s^2)^{\alpha_i(s)}P_{\alpha_i(s)}\left(-1-\frac{t}{2q_s^2}\right).$$

The quantity $R_i^{t_1}(s,t)$ is defined in an elementary sense

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[1] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).

[3] We ignore spin complications to simplify the discussion.

[4] N. Khuri, Phys. Rev. Letters **10**, 420 (1963) and Phys. Rev. **132**, 914 (1963), has proposed a similar decomposition using simple powers of s and t rather than Legendre functions of cosθ. The The Khuri form, however, turns out to have asymptotic properties that are unsuited to the strip approximation, as shown in the following paper [C. Edward Jones, Phys. Rev. **135**, B214 (1964)].

by formula (II.3) for $-1<\text{Re}\alpha_i(s)<0$ and otherwise by analytic continuation. The function $\alpha_i(s)$ is the position of the ith Regge pole and is assumed to be real analytic in the s plane cut from s_0 to $+\infty$. The function $\gamma_i(s)$ is the reduced residue (the actual residue divided by $q_s^{2\alpha_i(s)}$) and is assumed to have the same reality-analyticity properties as $\alpha_i(s)$. The terms $R_i^{t_1}(s,t)$ can be shown individually to satisfy the Mandelstam representation with double spectral functions asymptotic to $s=s_0$ and $t=t_1$. Similarly, $R_j^{s_1}(t,s)$ will be a sum of terms satisfying the Mandelstam representation, but here the asymptotes are $s=s_1$ and $t=t_0$.[5] In order to justify the choice (II.3) it is necessary to consider the asymptotic behavior of $\alpha_i(s)$ and $\gamma_i(s)$. This is done in the following paper.

The displacement of the t branch point from t_0 to $t_1(t_1>t_0)$ in $R_i^{t_1}(s,t)$ and of the s branch point from s_0 to s_1 in $R_j^{s_1}(t,s)$ facilitates the formulation of dynamical equations in the new form of strip approximation, as already discussed in Ref. 1 where the physical meaning of t_1 is explained. So long as one maintains in (II.2) the convergent double integral, the displacement in question merely changes the value of ρ_{st}', and one of the features of the new strip approximation is the assumption that this convergent integral is small.

The first step in our approximation scheme then is to represent the full amplitude as

$$A(s,t)\approx\sum_i\left[R_i^{t_1}(s,t)+\xi_iR_i^{u_1}(s,u)\right]+\sum_j\left[R_j^{s_1}(t,s)+\xi_jR_j^{u_1}(t,u)\right]+\sum_k\left[R_k^{s_1}(u,s)+\xi_kR_k^{t_1}(u,t)\right]\quad\text{(II.4)}$$

with only the leading trajectories being included and the sign factor $\xi_{i,j,k}$ being ±1 depending on the signature of the trajectory in question. Each of the six terms corresponds to a piece of the double spectral function that is dominant in a particular strip in the sense of Fig. 1. Explicit formulas for the double spectral functions corresponding to (II.4) are given below in Eq. (III.6).

We now list the obvious aspects in which the approximation (II.4) is satisfactory. First, it contains all the poles near the physical region with the correct residues, and if all selected trajectories stay to the right of $l=-1$ there are no spurious singularities with a strength to compete with poles. Near any important pole of s, in other words, for all values of t (or u) we are guaranteed accuracy; a corresponding statement also holds near poles in t or u. At low energies, in particular, we have at least the accuracy of the (many-level) Breit-Wigner formula in the physical resonance region for low angular momentum, whereas scattering for high angular momentum is controlled by the low-mass particles in the t and u channels in the manner by now experimentally verified.[6] The correct threshold behavior as a function of angular momentum is guaranteed by (II.4) as is the general analytic structure of partial-wave amplitudes.

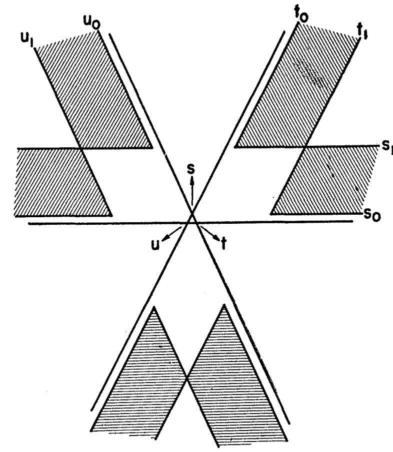

FIG. 1. The Mandelstam diagram, showing the strip regions where the double spectral functions are dominated by Regge poles.

What about high energies? If the only l singularities are simple poles, then as is well known (II.4) becomes asymptotically accurate for low momentum transfers as well as for individual partial waves. With branch points in l the situation is more complicated, but we know from empirical fits to experiment that the pole approximation at high energies does not go wildly wrong. In particular, it represents the experimental behavior of total cross sections rather well. The use of (II.4) therefore ensures a more satisfactory treatment of high energies than has been achieved in any pre-Regge dynamical calculations. It is the intermediate energies, i.e., near the edge of the strip, whose description is of dubious status. In particular, the formula (II.3) becomes logarithmically infinite at $t=t_1$ in violation of the unitarity condition in the t channel. This deficiency will be remedied in the second stage of our approximation scheme when we apply the unitarity condition in Sec. IV, but its presence in (II.4) forces us to remember that the sharp boundary for the strip is artificial.

Even though (II.4) does not satisfy unitarity exactly in any channel, we hope that the violation is minor except near the strip boundaries and that by explicit imposition of unitarity in the second step of our program a sensible, smooth connection between high and low energies across the boundary can be achieved. As a final argument in support of the plausibility of formula (II.4) we remark that it corresponds to the separation of the amplitude, familiar in classical nuclear physics, into "direct" and "indirect" scattering. In the s channel, for example, the terms R_j and R_k arising from crossed poles give the "direct" or "potential" scattering that dominates high angular momentum and high energies. The terms R_i represent "indirect" or "resonance" scattering and are important only for low angular momentum and low energies. From the dynamical

[5] In Ref. 1 a more complicated form than (II.3) was proposed for the contribution from a single pole. Both the old and the new forms seem physically acceptable.

[6] We refer here to what are usually called "peripheral" collisions.

standpoint, of course, the resonance scattering is "driven" by the potential.

III. THE GENERALIZED POTENTIALS

As a preliminary to step two of our scheme we introduce now two new amplitudes $A^{\pm}(s,z_s)$, each having a cut only for positive $\cos\theta = z_s$ when $s > s_0$. The Mandelstam representation for the original amplitude $A(s,z_s)$ can be written

$$A(s,z_s) = A_R(s,z_s) + A_L(s,z_s), \tag{III.1}$$

where

$$A_R(s,z_s) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{dt'}{t'-t(s,z_s)} D_t(t',s),$$
$$A_L(s,z_s) = \frac{1}{\pi}\int_{u_0}^{\infty} \frac{du'}{u'-u(s,z_s)} D_u(u',s), \tag{III.2}$$

D_t and D_u being the absorptive parts for the t and u channels, respectively. We then define

$$A^{\pm}(s,t) = A_R(s,z_s) \pm A_L(s,-z_s) \tag{III.3}$$

and observe that

$$A^{\pm}(s,t) = \frac{1}{\pi^2}\int\int ds'dt' \frac{\rho^{\pm}(s',t')}{(s'-s)(t'-t)}, \tag{III.4}$$

where

$$\rho^{\pm}(s,t) = \begin{cases} \rho_{st}(s,t) \pm \rho_{su}(s,t), & s > s_0, \\ -\rho_{tu}(t,u) \mp \rho_{tu}(u,t), & s < 0. \end{cases} \tag{III.5}$$

The even part in z_s of the original amplitude $A(s,z_s)$ coincides with the even part of $A^{+}(s,z_s)$ while the odd part coincides with the odd part of $A^{-}(s,z_s)$. Note, however, that A^{+} and A^{-} are individually neither even nor odd except when Bose statistics impose an additional constraint.

In the approximation (II.4) the various double spectral functions are given by

$$\rho_{st}(s,t) = \theta(t-t_1)\sum_i \mathrm{Im}\{R_i(t,s)\} + \theta(s-s_1)\sum_j \mathrm{Im}\{R_j(s,t)\},$$

$$\rho_{su}(s,u) = \theta(u-u_1)\sum_i \xi_i\, \mathrm{Im}\{R_i(u,s)\} + \theta(s-s_1)\sum_k \mathrm{Im}\{R_k(s,u)\}, \tag{III.6}$$

and

$$\rho_{tu}(t,u) = \theta(t-t_1)\sum_k \xi_k\, \mathrm{Im}\{R_k(t,u)\} + \theta(u-u_1)\sum_j \xi_j\, \mathrm{Im}\{R_j(u,t)\},$$

so that after some calculation we find

$$A^{\pm}(s,t) = \sum_i \left[R_i^{t_1}(s,t) \pm \xi_i R_i^{u_1}(s,t)\right] + \frac{1}{\pi}\int^{\infty} dt' \frac{V_s^{\pm}(t',s)}{t'-t}, \tag{III.7}$$

where the function $V_s^{\pm}(t,s)$ arising from the crossed poles is given by

$$V_s^{\pm}(t,s) = \frac{1}{\pi}\int_{s_1}^{\infty} \frac{ds'}{s'-s}\, \mathrm{Im}\{\sum_j R_j(s',t) \pm \sum_k R_k(s',t)\}$$
$$+ \frac{1}{\pi}\int_{u_1}^{\infty} \frac{du'}{u'-u}\, \mathrm{Im}\sum_j \xi_j R_j(u',t)$$
$$\pm \frac{1}{\pi}\int_{t_1}^{\infty} \frac{dt'}{t'-u}\, \mathrm{Im}\sum_k \xi_k R_k(t',t)$$
$$+ \theta(t-t_1)\sum_k \xi_k R_k(t,u)$$
$$\pm \theta(t-u_1)\sum_j \xi_j R_j(t,u), \tag{III.8}$$

and may be identified with the generalized potential defined by Chew and Frautschi.[7] The long-range parts of the potential including the poles in t are contained in the first two lines of (III.8).[8] The third line is a short-range part without poles.

It is possible to evaluate the crossed pole contributions to give

$$\frac{1}{\pi}\int^{\infty} dt' \frac{V_s^{\pm}(t',s)}{t'-t}$$
$$= \sum_j \left[R_j^{s_1}(t,s) + \xi_j R_j^{u_1}(t,u)\right]$$
$$\pm \sum_k \left[R_k^{s_1}(t,s) + \xi_k R_k^{t_1}(t,u)\right]$$
$$+ \frac{1}{\pi}\int_{t_1}^{\infty} dt' \left(\frac{1}{t'-t} \mp \frac{1}{t'-u}\right) \sum_k \xi_k R_k(t',u')$$
$$\pm \frac{1}{\pi}\int_{u_1}^{\infty} du' \left(\frac{1}{u'-t} \mp \frac{1}{u'-u}\right) \sum_j \xi_j R_j(u',t'), \tag{III.9}$$

with $s+u'+t' = s+u+t = \sum m^2$, the last two terms of (III.9) being odd functions of $\cos\theta_s$ for A^{+} and even functions for A^{-} and therefore not contributing to the physical amplitude A. In Ref. 1 these last terms were erroneously omitted. They correspond to short-range forces and contain no poles but are needed if the left-hand cut in $\cos\theta_s$ is to be completely removed. As will be seen in Sec. VI they are important in connection with asymptotic behavior. The essential feature of (III.9) as opposed to (III.8) is that for $t < 0$ and $s > s_0$ the pole positions and residues occur only with negative arguments and are correspondingly real. Thus the bootstrap calculation can be carried through with consideration only of l real and, in view of the Froissart limit,[9] $l \leqslant 1$.

[7] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).
[8] With proper attention to the definition of divergent integrals a bound state in the t or u channels can be shown on the basis of (III.8) to give the expected delta function in t.
[9] M. Froissart, Phys. Rev. **123**, 1053 (1961).

IV. THE N/D DYNAMICAL EQUATIONS FOR THE s CHANNEL

We assume as in Ref. 1 that inside each strip the two-body unitarity condition is adequate, leaving open the question of how many two-body reactions to include. For the s channel, if we suppress the $(\pm)$ superscript, the considerations described in Sec. III of Ref. 1 lead to the integral equation [(III.11) of the earlier paper]

$$N_l(s)=B_l^P(s)+\frac{1}{\pi}\int_{s_0}^{s_1}ds'\frac{B_l^P(s')-B_l^P(s)}{s'-s}\rho_l(s')N_l(s'),\quad \text{(IV.1)}$$

the amplitude for the lth partial wave in the s channel being given by

$$A_l(s)=q_s^{2l}B_l(s)=q_s^{2l}[N_l(s)/D_l(s)],\quad \text{(IV.2)}$$

where

$$D_l(s)=1-\frac{1}{\pi}\int_{s_0}^{s_1}ds'\frac{\rho_l(s')N_l(s')}{s'-s},\quad \text{(IV.3)}$$

ρ_l being the phase-space factor. The dynamics is then concentrated in the function $B_l^P(s)$, defined by

$$B_l^P(s)=B_l(s)-\frac{1}{\pi}\int_{s_0}^{s_1}ds'\frac{\operatorname{Im}B_l(s')}{s'-s},\quad l \text{ real},\quad \text{(IV.4)}$$

so as to contain the poles of the t and u channels but not those of the s channel. The latter arise from the zeros of $D_l(s)$.[10] The function $B_l^P(s)$ plays for the s channel somewhat the same role as the potential in a Schrödinger equation, but the analogy is not perfect. In particular $B_l^P(s)$ is not simply the partial-wave projection of formula (III.9), although such an approximation is often made and was advocated in Ref. 1. We shall see in the following section that $B_l^P(s)$ receives a contribution from the s-channel pole terms even though it contains itself no poles in s.

V. THE CALCULATION OF $B_l^P(s)$

The Froissart-Gribov definition of $B_l^\pm(s)$ for complex l can be given in either of two forms. The original form involves the discontinuity $D_t^\pm(t,s)$ of $A^\pm(s,t)$ in crossing the t cut,[11]

$$B_l^\pm(s)=\frac{1}{2\pi}\int_{t_0}^{\infty}\frac{dt}{q_s^{2l+2}}Q_l\left(1+\frac{t}{2q_s^2}\right)D_t^\pm(t,s),\quad \text{(V.1)}$$

while the second form, pointed out by Wong,[12] involves $A^\pm(s,t)$ itself[13]:

$$B_l^\pm(s)=-\frac{1}{2\pi}\int_{-\infty}^{0}\frac{dt}{q_s^{2l+2}}\left[\operatorname{Im}Q_l\left(1+\frac{t}{2q_s^2}\right)\right]A^\pm(s,t).\quad \text{(V.2)}$$

In the new strip approximation, $A^\pm(s,t)$ is given by formula (III.7) for which the corresponding t discontinuity is

$$D_t^\pm(t,s)=\sum_i R_i(t,s)[\theta(t-t_1)\pm\xi_i\theta(t-u_1)]+V_s^\pm(t,s).\quad \text{(V.3)}$$

Now $V_s^\pm(t,s)$ has no discontinuity in s for $s_0<s<s_1$, so from (IV.4) we see that in calculating $B_l^P(s)$ we are to take the entire generalized potential contribution, subtracting nothing away. The first term in (V.3) when substituted into (V.1) gives a function cut in s between $-\infty$ and s_0-t_1, due to the s discontinuity in $q_s^{-2l}Q_l(1+t/2q_s^2)$, and also cut between s_0 and $+\infty$ due to the s discontinuity of $R_i(t,s)$. The portion of the latter cut between s_1 and ∞ must be small if the strip approach makes sense, so when the cut between s_0 and s_1 is removed according to the prescription (IV.4) we have (taking $t_1=u_1$ and all masses equal)

$$B_l^{P\pm}(s)=V_{s,l}^\pm(s)+\frac{1}{\pi}\int_{-\infty}^{s_0-t_1}\frac{ds'}{s'-s}\times\sum_i\int_{t_1}^{s_0-s'}dt(1\pm\xi_1)R_i(t,s)\frac{P_l(-1-t/2q_{s'}^2)}{-4(-q_{s'}^2)^{l+1}}.\quad \text{(V.4)}$$

By $V_{s,l}^\pm(s)$ is meant the l-wave projection of the generalized potential contribution: e.g., formula (III.9) inserted in place of $A^\pm$ on the right-hand side of (V.2). By choosing this particular method only real pole positions and residues are encountered in the evaluation for $s_0<s<s_1$:

$$\begin{aligned}V_{s,l}^\pm(s)=&-\frac{1}{2\pi q_s^{2l+2}}\int_{-\infty}^{0}dt[\operatorname{Im}Q_l(1+t/2q_s^2)]\\&\times\left\{\frac{1}{\pi}\int_{s_1}^{\infty}\frac{ds'}{s'-s}[\sum_i R_j(s',t)\pm\sum_k R_k(s',t)]\right.\\&+\frac{1}{\pi}\int_{u_1}^{\infty}\frac{du'}{u'-u}\sum_i \xi_j(R_j(u',t)-R_j(u',t'))\\&\pm\frac{1}{\pi}\int_{t_1}^{\infty}\frac{du'}{u'-u}\sum_k \xi_k(R_k(u',t)-R_k(u',t'))\\&+\frac{1}{\pi}\int_{t_1}^{\infty}\frac{dt'}{t'-t}\sum_k \xi_kR_k(t',u')\\&\left.\pm\frac{1}{\pi}\int_{u_1}^{\infty}\frac{dt'}{t'-t}\sum_i \xi_jR_j(t',u')\right\}.\quad \text{(V.5)}\end{aligned}$$

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[10] Note that because $D_l(s)$ is real analytic in the s-plane cut between $s=s_0$ and $s=s_1$ the same will be true for any s-channel pole position $\alpha_i(s)$ or reduced residue $\gamma_i(s)$ calculated from a zero of $D_l(s)$, if multiple l poles are absent from $D_l(s)$.

[11] M. Froissart, Report to the La Jolla Conference on Theoretical Physics, June 1961 (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 and 1962 (1961) [English transl.: Soviet Phys.—JETP **14**, 478 and 1395 (1962)].

[12] D. Wong (private communication, 1962).

[13] We have written Eq. (V.2) for l real and $q_s^2>0$, the imaginary part of Q_l to be evaluated as the negative t axis is approached from above.

The expression (V.4) together with (V.5) is considerably more complicated than that for Ref. 1 but still contains the pole parameters only where they are real. The second term in (V.4), arising from the s-channel poles, had no counterpart in Ref. 1 and may not be of great importance for s inside the strip since the integral over ds' is entirely outside. Keeping this term, however, tends to alleviate the N/D conflict between threshold and asymptotic behavior that becomes severe for high values of l. Our N/D equations (IV.1) and (IV.3) in any event minimize this conflict by avoiding an integration to infinity, but the solutions for $l>1$, if examined as $s\to\infty$, necessarily violate the unitarity condition unless terms like those in (V.4) are included in $B_l^{P\pm}(s)$. If the partial-wave amplitude emerging from the N/D calculation were exactly of the form implied by the ansatz (II.4), the conflict with unitarity would be entirely removed by the extra terms. To the extent that input and output are roughly consistent, the conflict is alleviated.

In formula (V.5) integrations to $-\infty$ in t occur, whose convergence depends on the asymptotic behavior of the pole parameters. It is not expected in the strip approximation that this asymptotic behavior should be reliable, but unless the integrals in (V.5) are strongly convergent there will be important contributions from outside the strip that cast doubt on the consistency of the whole approach. Let us now consider, therefore, the behavior of pole parameters for large negative argument in connection with the evaluation of (V.5).

VI. ASYMPTOTIC BEHAVIOR OF THE POLE PARAMETERS

It is not difficult to show that as $t\to\infty$ for s fixed $R_j^{s_1}(t,s)$ behaves like $\gamma_j(t)t^{\alpha_j(t)}\ln^2 t$,[14] so this combination of factors should vanish for large t if the strip concept is to have any validity. Such a vanishing, furthermore, is required if the integrals appearing in the expressions (V.4) and (V.5) are to converge for all Re$l\geqslant 0$. The Froissart limit[9] guarantees that all poles retreat to the left of $l=1$ for negative t, so it will suffice to have $\gamma_j(t)$ decrease asymptotically at least as fast as t^{-1}.

As our denominator function $D_l(s)$ is constrained through (IV.3) to approach 1 as $s\to\infty$ for any finite $N_l(s)$, the position in the l plane of a zero of $D_l(s)$ for large s must approach an infinite fixed-l singularity of the numerator function $N_l(s)$. In particular, the numerator function may have fixed poles arising from the solution of Eq. (IV.1), which has been shown to be essentially Fredholm in character.[15] For nonrelativistic potential scattering Taylor has shown that there are no poles in $N_l(s)$ beyond those already appearing in the potential and that it suffices to analyze the fixed singularities of the potential (i.e., the Born approximation) in order to deduce the asymptotic behavior of Regge-pole parameters.[16] We have no such assurance in our case and in fact must expect Fredholm (dynamical) fixed poles in the numerator function. In particular there are neighborhoods in the complex l plane where the kernel of the integral equation (IV.1) is unbounded in normalization. The most apparent such neighborhoods are near the Gribov-Pomeranchuk fixed poles at $l=-1, -2, \cdots$ of formula (III.9) for the s-channel generalized potential. These poles necessarily occur in $B_l^P(s)$ through the first term of (V.4), a straightforward calculation showing that they cannot be canceled by the second term of this formula.[17] Near one of these poles the kernel of (IV.1) can achieve an almost arbitrary normalization without much change in the (s,s') dependence. It follows that an infinite number of eigenvalues of the homogeneous equation will be accessible. In other words, each fixed-l pole of the generalized potential will produce a swarm of Fredholm fixed-l poles in the numerator function, and each of the Fredholm fixed-l poles then will serve as a possible terminal point for a Regge trajectory. The novel feature of this situation is that our terminal points are dynamically determined and will vary according to the force strength.

Let us now examine the possible additional fixed-l singularities contained in formula (V.4) for $B_l^P(s)$. In the generalized potential as given by (III.9) there are two types of terms, corresponding to the two distinct double spectral regions in (III.5):

$$\text{(a)}\qquad \frac{1}{\pi}\int_{s_1}^{\infty} ds'\frac{R_j(s',t)}{s'-s}$$

$$\text{(b)}\qquad \frac{1}{\pi}\int_{t_1}^{\infty} dt'\left\{R_j(t',u')\left[\frac{1}{t'-t}\mp\frac{1}{t'-u}\right]\pm R_j(t',t)\frac{1}{t'-u}\right\}. \tag{VI.1}$$

The asymptotic behavior for large t determines the location of the leading singularity in the l plane. By assuming that $\gamma_j(t)\sim t^{-\epsilon_j}$ the leading singularity in (a) occurs at $l=\alpha_j(\infty)-\epsilon_j$. On the other hand, terms of the type (b) have the Gribov-Pomeranchuk pole at $l=-1$ for even signature and $l=-2$ for odd, as well as a singularity at $l=\alpha_j(\infty)-\epsilon_j$. The second term of (V.4) has its leading singularity at $l=\alpha_i(\infty)-\epsilon_i$.

Now suppose that[18] ϵ_i, $\epsilon_j>2$ so the contributions out-

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[14] This demonstration is given explicitly in the following paper by one of the authors (C. E. J.).

[15] G. F. Chew, Phys. Rev. **130**, 1264 (1963).

[16] John Robert Taylor, Ph.D. thesis, University of California, Berkeley, June 1963 (unpublished).

[17] It is the presence of energy cuts in the relativistic generalized potential that prevents a cancellation, as first noted by Gribov and Pomeranchuk, in *Proceedings of the 1962 International Conference on High Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 522. See also Phys. Letters **7**, 239 (1962).

[18] R. Serber, Phys. Rev. Letters **10**, 357 (1963), has pointed out that high-energy elastic-scattering cross sections appear to fall off as the inverse fifth power of momentum transfer squared. This would imply $\epsilon\approx 2.5$ for the Pomeranchuk trajectory.

side the strip are really small. Terms of the type (b) then dominate the t asymptotic behavior of the generalized potential at least for positive signature and correspondingly should play a controlling role in the asymptotic behavior of Regge poles. In particular, for positive signature we anticipate a cloud of fixed-l Fredholm poles in the numerator function to surround the point $l=-1$ (where there must be an essential singularity, as emphasized by Gribov and Pomeranchuk[17]), the maximum displacement of the poles from their "source" depending on the force strength. Assuming no trajectory intersections, the Fredholm pole standing farthest to the right must be the terminal point of the leading Regge trajectory, and without a numerical calculation all we can say about its position is that it must lie between $l=-1$ and $l=+1$.[19] Of course, once the possibility is raised that with very strong forces this terminal point may lie to the right of $l=0$, one is tempted to see here a means of avoiding the well-known awkwardness with the Pomeranchuk trajectory when the point $l=0$ is crossed at negative energy.

Next, what about the asymptotic behavior of reduced residues? In the following paper it is shown that when the leading singularity of the numerator function is a simple pole the reduced residue vanishes at least as fast as $1/s$. The possibility of a multiple pole in $N_l(s)$ is also discussed. It has not been possible to demonstrate as strong a tendency to vanish asymptotically as is indicated experimentally or as was assumed above, and if ϵ_j is actually equal to 1, the potential terms would have a fixed singularity at $l=\alpha_j(\infty)-1$ for both signatures which might be more important than the Gribov-Pomeranchuk singularity. The above arguments would not thereby be altered in any important way, but in any event there is no reason to trust our equations outside the strip. If the rate of change of $\gamma_i(s)$ near $s=0$ is successfully described we shall be satisfied.

VII. SUMMARY AND CONCLUSION

We have presented a set of dynamical equations suitable for bootstrap calculations with zero-spin external particles. The scattering amplitude is represented in two alternative ways, the pole superposition (II.4) and the N/D prescription of Sec. IV, neither of which is exact but both of which are supposed to be reasonably accurate at low energies and low angular momentum where bound states and resonances occur. The bootstrap calculation consists of a matching of the pole parameters in the two forms for real $l\leqslant 1$ and low energies. The pole superposition then gives the high angular-momentum components at low energy and hopefully the low momentum-transfer behavior at high energy.

The spirit of this paper is the same as that of Ref. 1, and the N/D prescription has not been changed in any way from that of the earlier paper. We have proposed here, however, an explicit and simple expression for the pole superposition that conforms term by term to the Mandelstam representation. The clarity thereby achieved has allowed the correction of an error in Ref. 1 involving the "third" double spectral region. We are also proposing now to augment the "input" function $B_l^P(s)$ for the N/D equations by a contribution from the direct-channel poles.

An analysis of our bootstrap equations has revealed two physically important features absent in ordinary potential scattering (and which do not accord with conjectures made in Ref. 1): (a) The terminal point for our Regge trajectories is dynamically determined and for strongly attractive forces may lie to the right of $l=0$.[20] (b) Our reduced residues vanish for large energy at least as fast as $1/s$. Both these features have immediate relevance to the problem of fitting high-energy data with Regge poles.

There remains the problem raised by Mandelstam of cuts in angular momentum.[21] This difficulty has had no chance to arise here because we have not attempted explicitly to impose unitarity beyond the two-body region. Conceding the correctness of Mandelstam's conclusion, there is still room for belief that our bootstrap scheme is sensible if the cuts are weak in importance compared to the poles. In energy and momentum-transfer variables the dominant role played by poles has been *the* striking feature of strong-interaction physics; the same may well be true for angular momentum.

Put another way, in Ref. 1 it was pointed out that experimentally the bulk of resonance decay seems to occur in two-body channels if unstable particles are considered. This circumstance, coupled with the assumption that stable and unstable particles eventually will achieve equivalent status in the dynamics, suggests that conclusions based on the two-body unitarity condition have a wide range of validity. Our approximation scheme can handle any finite number of two-body reactions, with the choice of the parameter s_1 depending on how many channels are included. Hopefully, when a sufficiently large number of channels is incorporated into the N/D calculation, the precise value of s_1 will become unimportant. Were that to happen, the goal of a parameter-free dynamics would have been achieved.

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[19] The constraint to lie to the left of $l=+1$ is not built explicitly into our equations but, as explained in reference 1, is to be imposed separately.

[20] The latter circumstance would not invalidate our whole program because there will still be regions of energy (perhaps on unphysical sheets) where the pole retreats to the left and allows the function (II.3) to be defined.

[21] S. Mandelstam, Nuovo Cimento **30**, 1148 (1963).